VaR-based portfolio optimization on the Stock Exchange in Warsaw

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Abstract

An empirical analysis is performed of portfolios selected with respect to value at risk on the Stock Exchange in Warsaw between the years 1991–2008. Value at risk is estimated under normal or any constant distribution of rates of return on time intervals. The methods examined maximize expected portfolio return putting upper bound on value at risk. It is shown that with probabilities much higher than the level of value at risk true returns are lower than their bounds.

1 Introduction

Let X be a random variable. Value at risk at a level $\alpha \in (0,1)$ is the $(1-\alpha)$-quantile of the distribution of $-X$ [4, p. 168], i. e.

$$\begin{aligned} VaR_\alpha(X) &= \inf\{x \in \mathbf{R} \colon P(-X \leq x) \geq 1-\alpha\} \\ &= -\sup\{x \in \mathbf{R} \colon P(X < x) \leq \alpha\}. \end{aligned}$$

If X represents a portfolio return, the probability that it is lower than $-VaR_\alpha(X)$ is not grater than α. Thus, investors are interested in small value at risk.

Given n assets whose rates of return $r = (r_1, \dots, r_n)$ are random variables, the rate of return of the portfolio $w = (w_1, \dots, w_n) \in \mathbf{R}^n$, $w_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n w_i = 1$, can be expressed as $X = wr^T$.

If r is distributed $N(\mu, \Sigma)$, X is distributed $N(w\mu^T, w\Sigma w^T)$ and [4, p. 173]

$$VaR_\alpha(X) = q\sqrt{w\Sigma w^T} - w\mu^T, \tag{1}$$

where $q = \Phi^{-1}(1-\alpha)$ and Φ is the cumulative distribution function of $N(0,1)$. Portfolio optimization problem can take the form:

$$\begin{array}{lll} \text{maximize} & w\mu^T & \\ \text{subject to} & q\sqrt{w\Sigma w^T} - w\mu^T & \leq v \\ & \sum_{i=1}^n w_i & = 1 \\ & w_i & \geq 0 \end{array} \tag{2}$$

The first and second derivatives of (1) are, respectively,

$$\frac{q}{\sqrt{w\Sigma w^T}}w\Sigma - \mu \quad \text{and} \quad \frac{q}{\sqrt{w\Sigma w^T}}\left(\Sigma - \frac{\Sigma w^T w\Sigma}{w\Sigma w^T}\right).$$

Applying the Cauchy-Schwarz inequality to the scalar product $\langle v, w\rangle = v\Sigma w^T$ it can be shown that the second derivative is nonnegative definite (provided that $\alpha < 1/2$), so (2) is a problem of convex optimization [3, §13.3].

The purpose of this paper is to evaluate the fraction of crosses of value at risk for optimal portfolios on the Stock Exchange in Warsaw from April 1991 until the end of 2008.

2 Methods

Apart from the level α, the experiment had three additional parameters: the length of investment period f, the sample horizon h and the preferred annual rate of return p which determined the upper bound of value at risk $v = -pf/360$. Values considered were $h = 90,\ 180$ days, $f = 1,\ 7,\ 30,\ 360$ days, $\alpha = 0.01,\ 0.02,\ 0.05$, $p = 0.05,\ 0.10$.

For each set of h, f, α, p and for each day d between 16 April 1991 and 31 December 2008, a sample of all assets was selected which were quoted on each session of the Stock Exchange between $d-f-h$ and $d-1$ and on days d and $d+f$, with $d+f$ before 1 May 2009. If the sample had less than 2 assets or less than 30 observed rates of return from before d, it was rejected. Data preceding d were used to estimate parameters μ and Σ by the method of moments [2, p. 379] and the optimal portfolio was selected solving the problem (2). Data from the days d and $d+f$ were used to calculate the true rate of return achieved by the selected portfolio. If it proved less than $-VaR_\alpha(X)$, value of risk was crossed [1, p. 466].

3 Results

The number of experiments for each set of parameters ranged from 1769 to 2811, depending on the sample horizon and the length of investment period. A number of experiments had no feasible solution due to the first constraint in (2); in these cases the method did not admit any portfolio to possess value at risk low enough. In particular, daily investments were forbidden for half-yearly horizon. The numbers of experiments for which a portfolio could be created are given in table 1.

Relative frequencies $\hat{\alpha}$ of crosses are given in table 2. As the statistic $\sqrt{n}(\hat{\alpha} - \alpha)/\sqrt{\alpha(1-\alpha)}$, where n denotes the respective sample size from table 1, is asymptotically distributed $N(0, 1)$ [5, §2.7], each hypothesis that

Table 1: Numbers of experiments for which portfolio (2) existed.

h	α	$p = 0.05$				$p = 0.10$			
		f				f			
		1	7	30	360	1	7	30	360
	0.01	111	749	1391	1742	98	719	1350	1721
90	0.02	135	859	1469	1751	121	804	1420	1735
	0.05	184	1031	1566	1769	172	971	1535	1763
	0.01	0	113	791	1608	0	96	736	1551
180	0.02	0	133	845	1691	0	125	815	1613
	0.05	0	240	946	1745	0	225	880	1717

Table 2: Relative frequencies of crosses of value at risk for problem (2).

h	α	$p = 0.05$				$p = 0.10$			
		f				f			
		1	7	30	360	1	7	30	360
	0.01	0.41	0.47	0.55	0.80	0.41	0.48	0.57	0.80
90	0.02	0.41	0.45	0.55	0.82	0.37	0.46	0.56	0.83
	0.05	0.43	0.47	0.58	0.84	0.41	0.47	0.59	0.85
	0.01	–	0.29	0.50	0.69	–	0.32	0.51	0.72
180	0.02	–	0.32	0.51	0.72	–	0.31	0.52	0.75
	0.05	–	0.38	0.54	0.75	–	0.38	0.55	0.76

the frequency of crosses is equal to the level of value at risk can be rejected at significance level as small as 10^{-90}.

4 Discussion

4.1 Nonparametric approach

Assumption of normality of returns made in section 1 is not necessary for optimizing portfolio. Assuming no particular probability distribution function and basing on m historical observations $r_i = (r_{i1}, \ldots, r_{in})$, $i = 1, \ldots, m$, $VaR_\alpha(X)$ can be estimated as the $(1-\alpha)$-quantile of the empirical distribution function of the variable $-X$, i. e. as [5, §2.1, §2.3]

$$\inf \left\{ x \in \mathbf{R} \colon \sum_{i=1}^{m} I(-wr_i^T \le x) \ge m(1-\alpha) \right\},$$

where $I(-wr_i^T \le x)$ takes on value 1 if $-wr_i^T \le x$ and 0 otherwise. Using the sample value at risk results in the problem

$$\begin{array}{lll} \text{maximize} & w\mu^T & \\ \text{subject to} & \sum_{i=1}^{m} I(-wr_i^T \le v) & \ge \quad m(1-\alpha) \\ & \sum_{i=1}^{n} w_i & = \quad 1 \\ & w_i & \ge \quad 0 \end{array} \tag{3}$$

Table 3: Numbers of experiments for which portfolio (3) existed.

h	α	$p=0.05$, f = 1	7	30	360	$p=0.10$, f = 1	7	30	360
	0.01	689	1571	1823	1769	656	1530	1800	1762
90	0.02	758	1823	1823	1769	729	1735	1800	1762
	0.01	22	515	1351	1787	14	454	1306	1671
180	0.02	89	590	1466	1787	64	542	1373	1757

Table 4: Relative frequencies of crosses of value at risk for problem (3).

h	α	$p=0.05$, f = 1	7	30	360	$p=0.10$, f = 1	7	30	360
	0.01	0.46	0.48	0.56	0.84	0.48	0.48	0.58	0.84
90	0.02	0.46	0.48	0.56	0.84	0.48	0.48	0.58	0.84
	0.01	0.50	0.49	0.52	0.72	0.57	0.49	0.54	0.78
180	0.02	0.38	0.48	0.54	0.74	0.47	0.49	0.56	0.75

The first condition in (3) states that the number of observations for which $-wr_i^T \le v$ must not be less than $m(1-\alpha)$. It is enough to limit this number to $k = \lceil m(1-\alpha) \rceil$. For each k-element subset $\{i_1, \ldots, i_k\}$ of the set $\{1, \ldots, m\}$ the following linear programming problem should be solved

$$\begin{array}{lll} \text{maximize} & w\mu^T & \\ \text{subject to} & -wr_{i_1}^T & \le v \\ & \cdots & \\ & -wr_{i_k}^T & \le v \\ & \sum_{i=1}^n w_i & = 1 \\ & w_i & \ge 0 \end{array} \tag{4}$$

and the maximum solution over these subsets should be selected.

Even though the number of problems (4) is $\binom{m}{k}$, as the number m of observed rates of return was at most 126, it was possible to solve them for $\alpha = 0.01,\ 0.02$. It can be seen from tables 3 and 4 that there is no important difference between the two approaches.

4.2 Value at risk in presence of commission

Taking into account a commission rate $0 \le c < 1$ on each transaction, the rate of return of the portfolio can be rewritten as $X = C\left(1 + wr^T\right) - 1$, where $C = (1-c)/(1+c)$.[1] Since $VaR_\alpha(X) = qC\sqrt{w\Sigma w^T} - C(1 + w\mu^T) + 1$, the

[1] Investing capital K one assigns $K/(1+c)$ for assets and $cK/(1+c)$ for commission. After buying $(K/(1+c))w_i/p_i$ assets at a price p_i, and then selling them at a price p_i' one must assign $c\sum(K/(1+c))w_i p_i'/p_i$ for commission. The two operations yield the return $C\sum w_i(p_i'/p_i) - 1 = C(1 + \sum w_i r_i) - 1$.

problem equivalent to (2) is

$$\begin{array}{lllll} \text{maximize} & C(1+w\mu^T)-1 & & & \\ \text{subject to} & q\sqrt{w\Sigma w^T}-w\mu^T & \leq & 1+(v-1)/C & \\ & \sum_{i=1}^{n} w_i & = & 1 & \\ & w_i & \geq & 0 & \end{array} \qquad (5)$$

It is worth noting that for $v < 1$ the solution of (2) may be unfeasible for (5).

For both model (5) and its counterpart for (3), the greater the rate of commission, the smaller the number of feasible portfolios and the greater the frequency of crosses.

5 Conclusion

It has been proved that true rates of return of portfolios selected by maximizing expected rate of return at bounded value at risk frequently cross their upper bounds. Unquestionably relative frequencies of such crosses fail to meet the definition of value at risk. It may reasonably be doubted whether rates of return of portfolios are independent and possess constant distributions within intervals. Similar evidence was given in [6] about portfolio selection based on the Markowitz portfolio theory.

References

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