# On the distributions of stock price changes on the Stock Exchange in Warsaw

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## Abstract

The paper examines series of differences of logarithms of prices of 24 stocks quoted on the Stock Exchange in Warsaw in 1991–1998. For some of the stocks, first differences seem to be independent random variables distributed according to a class of uncontinuous laws. For the others, higher differences seem to follow a number of autoregressive processes with Laplace distributed disturbances.

**Keywords**: stable distribution, Laplace distribution, integrated autoregressive process.

# 1 Introduction.

It is often assumed that changes of prices of stocks are values taken on by random variables. The reasonable question is whether or not changes of prices of each stock are distributed according to the same law. After Bachelier [1] and Mandelbrot [13], it is usually accepted that if such a law exists, it belongs to the family of normal distributions or to the wider class of stable distributions. Both hypotheses have been carefully considered and extensively tested (see [4], [11], [10], [16], [14], [9], [12] for example). Most of empirical works refer to stocks quoted on the New York Stock Exchange. Some of the results confirm normality of stock price changes, while the others confirm stability. The purpose of this paper is to study distributions of stock price changes on the Stock Exchange in Warsaw since its opening in 1991 until the end of 1998.

The Stock Exchange in Warsaw is a small exchange in an emerging economy with 11 billion dollars turnover in 1998 and 198 stocks quoted on the last session that year. Every stock is quoted once a day at the price that maximizes the trading volume. However, if the maximizing price is greater (or lower) than the previous-day price by more than 10 per cent of the latter, the current price is set to 110 (or 90) per cent of the previous-day price. Nonetheless, absolute values of about 0.5 per cent of daily rates of return in the period of 1991–1998 exceeded 10 per cent. Dividends are paid once a year, the average dividend yield in 1998 being 0.9 per cent. Consequently, prices considered hereafter have not been corrected with respect to dividends.

For any sequence of numbers $\{x_t\}_{t=1}^{\infty}$ we define the progressive difference operator $\Delta$, which assigns to $\{x_t\}$ the sequence $\{\Delta x_t\}_{t=1}^{\infty}$, $\Delta x_t = x_{t+1} - x_t$, and its powers:

$$\Delta^0 x_t = x_t,$$
$$\Delta^d x_t = \Delta\left(\Delta^{d-1} x_t\right), \quad d = 1, 2, 3, \ldots$$

Let $\{x_t\}_{t=1}^{T}$ be a series of natural logarithms of stock prices. We will be examining the series

$$\left\{\Delta^d x_t\right\}_{t=1}^{T-d}, \quad d = 1, \ldots, 6, \tag{1}$$

for the first 24 stocks quoted on the exchange. The numbers of terms $T$ vary from 1140 to 1474 and are given in table 1.

## 2 Estimation and testing.

In the next section we examine stability of price changes. A necessary and sufficient condition for a random variable to be stably distributed is that its characteristic function is of the form [8]

$$\varphi(t) = \exp\{iat - |bt|^\alpha[1 + i\beta \operatorname{sgn}(t)\omega(|t|, \alpha)]\}, \tag{2}$$

where $\alpha, \beta, a, b$ are real constants, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $b \geq 0$, $\operatorname{sgn}(t)$ is equal to $-1$, 0 or 1 depending on whether $t$ is less, equal to or greater than 0, whereas

$$\omega(|t|, \alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ (2/\pi)\ln|t| & \text{otherwise.} \end{cases}$$

The parameter $\alpha$ is the characteristic exponent, $\beta$ is an index of skewness, $a$ is the location parameter and $b$ is the scale parameter. If $\beta = 0$, the distribution is symmetric with respect to $a$. If $\alpha > 1$, $a$ is the expected value of the distribution. Values of the cumulative distribution function and its derivatives may be evaluated on the basis of the formulae given by Zolotarev [17].

In agreement with the theorem proved in [3], if the parameters $\theta = (\alpha, \beta, a, b)$ of a stable distribution with characteristic function (2) belong to the set

$$\{(\alpha, \beta, a, b) : \alpha \in (\epsilon, 1) \cup (1, 2), |\beta| < 1, b > 0\} \subset \mathbf{R}^4, \tag{3}$$

where $\epsilon > 0$ is arbitrary small, then the sequence $\hat{\theta}_n$ of maximum likelihood estimators based on the first $n$ independent observations is consistent and asymptotically normal with mean $\theta$ and covariance matrix $n^{-1}I_\theta^{-1}$, where $I_\theta$ is the Fisher information matrix. When estimating parameters of stable distributions for the series (1), we were maximizing the logarithm of likelihood

$$l(\theta) = \sum_{t=1}^{T-d} \ln f(\Delta^d x_t; \alpha, \beta, a, b),$$

where $f$ denotes the probability density corresponding with (2), on the parameter space (3) by the method of Broyden, Fletcher, Goldfarb and Shanno [15, chapt. 5.11]. As starting points for iterative processes, we took the 1/2-truncated mean as an estimator of $a$ and estimators of Fama and Roll [5] for symmetric stable distributions as estimators of $\alpha$, $\beta$ and $b$. A process was succesfull if after less than 40 iteration steps absolute values of all derivatives of $l$ with respect to parameters were less than $10^{-3}$ and the Fisher information matrix was positively defined.

In section 4 we estimate parameters of autoregressive models

$$\Delta^d x_t = \varphi_0 + \varphi_1 \Delta^d x_{t-1} + \ldots + \varphi_p \Delta^d x_{t-p} + \xi_t, \quad t = 1, \ldots, T - d, \tag{4}$$

for $p, d = 1, \ldots, 6$, where $\varphi_0, \varphi_1, \ldots, \varphi_p$ are real parameters and

$$\xi_1, \ldots, \xi_{T-d} \tag{5}$$

are independent identically Laplace distributed random variables. Since probability density of Laplace distribution is given by

$$\frac{1}{2b} \exp\{-|x-a|/b\},$$

where $a$ and $b > 0$ are median and absolute deviation, respectively, the maximum likelihood estimates of parameters of autoregressive models are the least absolute error estimates, which may be found by simplex method.

To test independence of random variables, we make use of the following sign test. Let

$$X_1, \ldots, X_n, \quad n \geq 1, \tag{6}$$

be a sequence of identically distributed random variables. The distribution of the number of runs of negative and non-negative elements in this series, conditional on the number of elements of each kind, under assumption that they are independent, may be found in [6, chapt. II.11]. Let $p_l$ and $p_g$ be the probabilities that the number of runs is not greater and not less, respectively, than that in the series. If $p_l \leq \alpha/2$ or $p_g \leq \alpha/2$, the null hypothesis that (6) are independent should be rejected at the significance level $\alpha$.

We also employ the following $\chi^2$ and Kolmogorov-Smirnov goodness-of-fit tests. Let (6) be a sequence of independent identically distributed random variables possessing continuous cumulative distribution function $F(x;\theta)$, $\theta = (\theta_1, \ldots, \theta_m)$, and let $F_n(x)$ be the sample cumulative distribution function of the sample (6). If $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_m)$ is the maximum likelihood estimate of $\theta$ and

$$-\infty = a_0 < a_1 < \ldots < a_{r-1} < a_r = +\infty, \quad r = \lfloor \sqrt{n} \rfloor,$$

are real numbers such that $F(a_k; \hat{\theta}) - F(a_{k-1}; \hat{\theta}) = 1/r$, $k = 1, \ldots, r$, then the statistic

$$\chi^2 = rn \sum_{k=1}^{r} [F_n(a_k) - F_n(a_{k-1})]^2 - n$$

has a limiting $\chi^2$ distribution with $f = r - m - 1$ degrees of freedom. If

$$D_n = \sup_{x \in \mathbf{R}} |F_n(x) - F(x; \hat{\theta})|$$

is the Kolmogorov-Smirnov distance, the statistic $\sqrt{n} D_n$ has a limiting Kolmogorov distribution. Thus, as $n$ is large, if $p_\chi = P(\chi_f^2 \geq \chi^2) \leq \alpha$ or $p_K = P(K \geq \sqrt{n} D_n) \leq \alpha$ with $\chi_f^2$ and $K$ having respectively $\chi^2$ distribution with $f$ degrees of freedom and Kolmogorov distribution we should reject the null hypothesis that the true distribution of (6) is of the form $F(x; \theta)$ at the significance level $\alpha$.

To simplify computations when dealing with residuals from autoregressive models, we consider them to be independent, identically distributed random variables. In other words, we consider estimates to be the true values of unknown parameters.

# 3 Independence.

Table 1 lists values of $p_l$ and $p_g$ for these series (1) for which the hypothesis of independence may not be rejected at the significance level $10^{-10}$. It can be seen that only all 24 series for $d = 1$ are included. It seems that only first differences for PROCHNIK, SOKOLOW, VISTULA, WEDEL, WOLCZANKA and ZYWIEC may be independent. This gives an idea of how strongly the data contradict the hypothesis.

Unlike for the first differences, rejection of the null hypothesis for the next differences occurs due to small values of $p_g$. For $d > 1$, the smallest value of $p_d$ is $1 - 6 \cdot 10^{-12}$ and the greatest value of $p_g$ is $9 \cdot 10^{-12}$. Thus, if we used only one-sided version of the test, in no case could we reject the hypothesis of independece for $d > 1$.

In spite of the above results, we have found the maximum likelihood estimates of parameters of the the normal, stable and Laplace distributions for series (1).

The $\chi^2$ test rejected the hypothesis of normality for series (1) at the significance level as small as $10^{-6}$. The number $n(d)$ of successful estimations of parameters of stable distributions for each $d$ is given in table 2. The next columns of the table contain, for some values of $\alpha$, numbers $n(d, \alpha)$ of successfully estimated series (1), for which the hypothesis of stability may not be rejected by neither $\chi^2$ nor Kolmogorov-Smirnov goodness-of-fit test at the significance level $\alpha$. Results for Laplace distribution are given in the second section of table 2 in the same manner. Since $n(1, 10^{-4}) = 0$ for both stable and Laplace distributions, we reject the hypotheses that the first differences are distributed according to any of these laws.

Figure 1 presents a typical sample cumulative distribution function of $\Delta x_t$. A point of discontinuity may be observed at 0 as about 11 to 20 per cent of values of each series are exactly equal to 0. Furthermore, for each series there exists a quite long neighbourhood of 0, say $U$, such that $\Delta x_t \notin U$. We do not analyse this distribution further on. Figures 2–4 present the three approximations to the sample distribution function.

Let $L_{\mathcal{N}}$, $L_{\mathcal{S}}$ and $L_{\mathcal{L}}$ denote the maximized likelihood for respectively normal, stable and Laplace families of distributions. Let us assume that each series (1) is drawn from one of these distributions with the same probability. Then, the probability of drawing from Laplace distribution, conditional on the sample, is

$$\frac{L_{\mathcal{L}}}{L_{\mathcal{N}} + L_{\mathcal{S}} + L_{\mathcal{L}}}.$$

For all series (1), $d = 1$, for which all three likelihoods were known, these statistics are greater than $1 - 6 \cdot 10^{-15}$. Thus, if any series $\Delta x_t$ was drawn from one of the three laws, it was rather drawn from the Laplace law.

Comparing results for normal, stable and Laplace distributions, we may found that the distance between the sample cumulative distribution function and the estimated distribution is most frequently minimized by Laplace distribution.

# 4 Dependence.

In this section we consider the class of autoregressive models of the form (4). It may easily be shown that the common distribution of disturbances (5) does not belong to the class of normal distributions. Namely, after estimating models (4) for series (1) as if disturbances (5) were normally distributed by the method of maximum likelihood and applying the sign test and the $\chi^2$ goodness-of-fit test to the series of residuals we may found that there is no series for which simultaneously $p_l > 10^{-5}$, $p_g > 10^{-5}$ and $p_\chi > 10^{-5}$. Moreover, it has been shown in [7] that processes (1) do not belong to the class of ARMA, GARCH and ARMA-GARCH(1,1) models of low orders with normally distributed disturbances.

Situation becomes entirely different if we assume that distributions of disturbances (5) are elements of the class of Laplace distributions.

Table 3 presents numbers $n(d, \alpha)$ of series (1) for which hypotheses of independent and identically Laplace distributed disturbances (5) may not be rejected by neither sign nor goodness-of-fit tests at the significance level $\alpha$. For $d = 2$ and $\alpha = 0.01$, there exists an appropriate model for each stock except just for WOLCZANKA and ZYWIEC. Numerical results of testing residuals for all models (4) with $p_l$, $p_g$, $p_\chi$ and $p_K$ greater than 0.1 are given in table 4.

# 5 Conclusion.

It seems that stocks on the Stock Exchange in Warsaw may be classified into three groups:

**6 stocks,** for which first differences of logarithms of prices are independent identically distributed random variables with unknown uncontinuous distributions,

**13 stocks,** for which there exist $d > 1$ such that d-th differences of logarithms of prices are autoregressive processes of order $p$ for some $p \geq 1$ with Laplace distributed disturbances (in most cases $d = 2$ or $d = 3$),

**5 stocks,** for which existence of such autoregressive integrated processes is uncertain.

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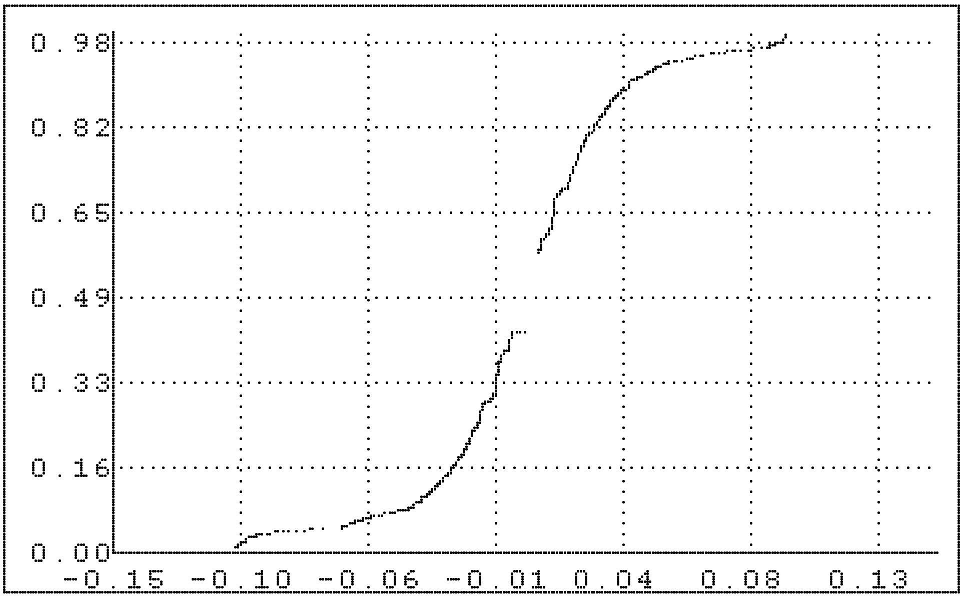


Figure 1: Sample cdf. of first differences of logarithms of prices for VISTULA.

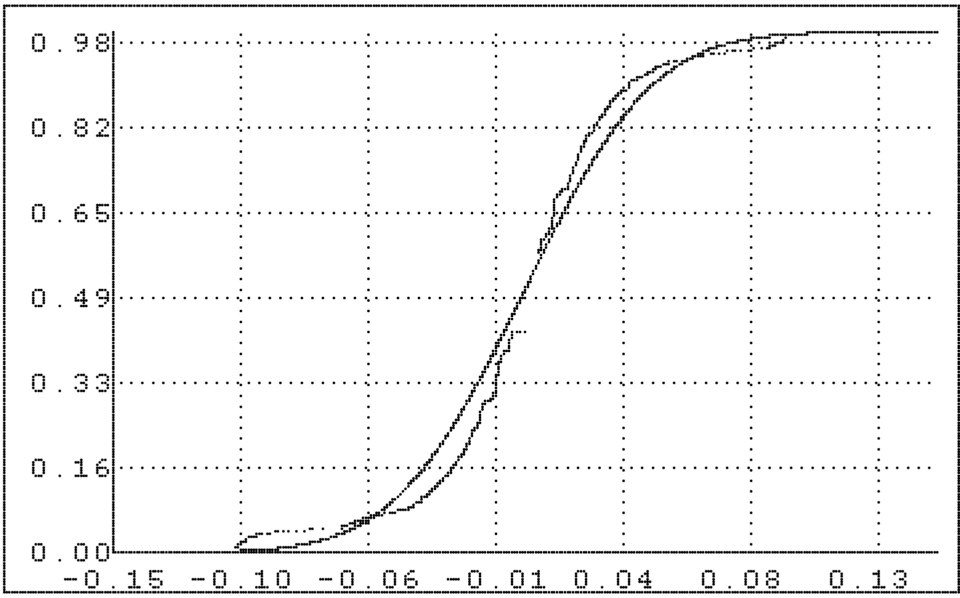


Figure 2: Sample cdf. of first differences of logarithms of prices for VISTULA and cdf. of fitted normal distribution.

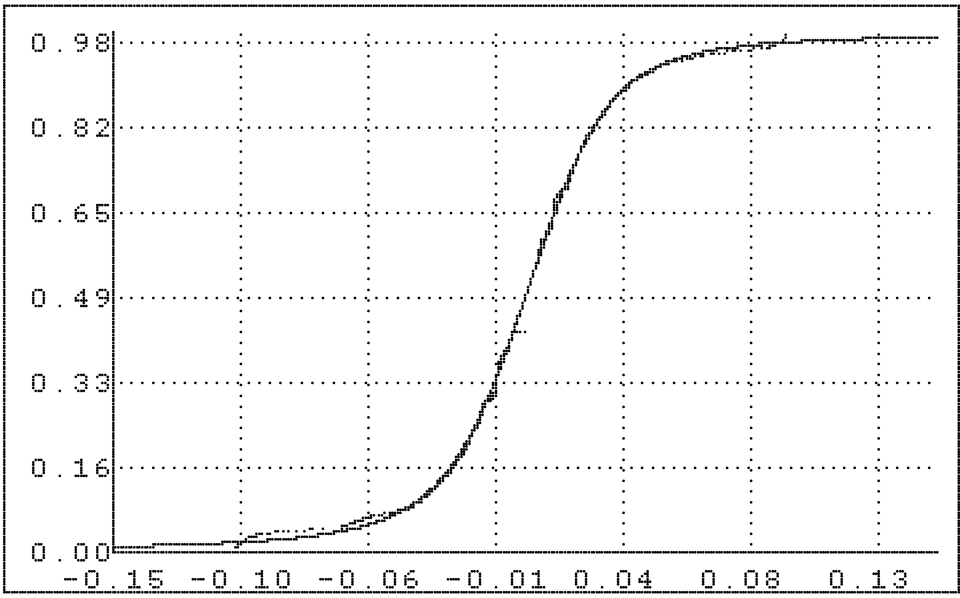


Figure 3: Sample cdf. of first differences of logarithms of prices for VISTULA and cdf. of fitted stable distribution.

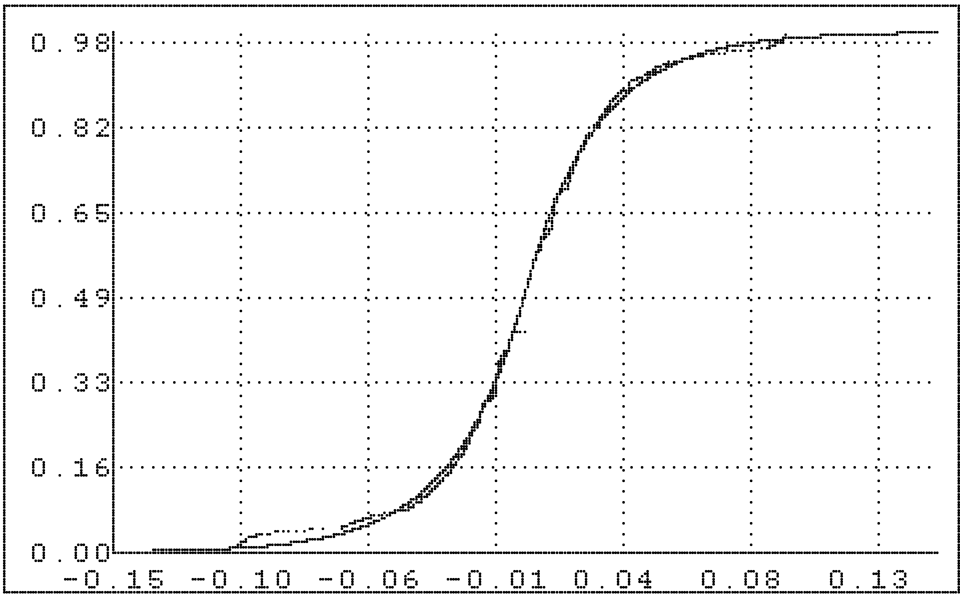


Figure 4: Sample cdf. of first differences of logarithms of prices for VISTULA and cdf. of fitted Laplace distribution.

Table 1: Sign test. Critical levels greater than $10^{-10}$, $d = 1$.

| STOCK | $T$ | $p_l$ | $p_g$ |
|---|---|---|---|
| BIG | 1372 | 0.0070 | 0.9940 |
| BRE | 1361 | 0.0000 | 1.0000 |
| BSK | 1176 | 0.0000 | 1.0000 |
| EFEKT | 1237 | 0.0114 | 0.9903 |
| ELEKTRIM | 1413 | 0.0000 | 1.0000 |
| EXBUD | 1474 | 0.0000 | 1.0000 |
| IRENA | 1429 | 0.0000 | 1.0000 |
| KABLE | 1472 | 0.0059 | 0.9950 |
| KROSNO | 1474 | 0.0002 | 0.9998 |
| MOSTALEXP | 1398 | 0.0042 | 0.9965 |
| MOSTALWAR | 1218 | 0.0014 | 0.9989 |
| OKOCIM | 1425 | 0.0004 | 0.9997 |
| POLIFARBC | 1283 | 0.0018 | 0.9985 |
| PROCHNIK | 1473 | 0.3748 | 0.6448 |
| RAFAKO | 1140 | 0.0000 | 1.0000 |
| SOKOLOW | 1246 | 0.1383 | 0.8744 |
| SWARZEDZ | 1464 | 0.0021 | 0.9982 |
| TONSIL | 1469 | 0.0093 | 0.9920 |
| UNIVERSAL | 1385 | 0.0001 | 1.0000 |
| VISTULA | 1224 | 0.3848 | 0.6379 |
| WBK | 1265 | 0.0004 | 0.9997 |
| WEDEL | 1257 | 0.1025 | 0.9083 |
| WOLCZANKA | 1462 | 0.1321 | 0.8788 |
| ZYWIEC | 1445 | 0.3056 | 0.7126 |

Table 2: Numbers of series which conform with a stable law or a Laplace law.

| $d$ | $n(d)$ | $n(d, 10^{-4})$ | $n(d, 10^{-3})$ | $n(d, 0.01)$ | $n(d, 0.05)$ | $n(d, 0.1)$ |
|---|---|---|---|---|---|---|
| | | | Stable law | | | |
| 1 | 13 | 0 | 0 | 0 | 0 | 0 |
| 2 | 17 | 0 | 0 | 0 | 0 | 0 |
| 3 | 20 | 0 | 0 | 0 | 0 | 0 |
| 4 | 22 | 13 | 11 | 7 | 5 | 3 |
| 5 | 19 | 17 | 12 | 9 | 6 | 5 |
| 6 | 22 | 15 | 11 | 10 | 6 | 4 |
| | | | Laplace law | | | |
| 1 | 24 | 0 | 0 | 0 | 0 | 0 |
| 2 | 24 | 0 | 0 | 0 | 0 | 0 |
| 3 | 24 | 9 | 3 | 2 | 1 | 1 |
| 4 | 24 | 23 | 21 | 17 | 11 | 9 |
| 5 | 24 | 23 | 22 | 21 | 19 | 18 |
| 6 | 24 | 23 | 22 | 22 | 20 | 16 |

Table 3: Numbers of series which conform with an autoregressive model with Laplace distributed residuals.

| $d$ | $n(d)$ | $n(d, 10^{-4})$ | $n(d, 10^{-3})$ | $n(d, 0.01)$ | $n(d, 0.05)$ | $n(d, 0.1)$ |
|---|---|---|---|---|---|---|
| 1 | 144 | 0 | 0 | 0 | 0 | 0 |
| 2 | 144 | 115 | 95 | 69 | 38 | 23 |
| 3 | 144 | 94 | 70 | 42 | 12 | 4 |
| 4 | 144 | 58 | 38 | 12 | 3 | 0 |
| 5 | 144 | 35 | 18 | 8 | 1 | 1 |
| 6 | 144 | 7 | 2 | 0 | 0 | 0 |

Table 4: Autoregressive models with Laplace distributed residuals, critical levels greater than 0.1.

| STOCK | $d$ | $p$ | $p_l$ | $p_g$ | $p_\chi$ | $p_K$ |
|---|---|---|---|---|---|---|
| BSK | 5 | 6 | 0.8795 | 0.1326 | 0.5960 | 0.9496 |
| EFEKT | 3 | 4 | 0.8670 | 0.1456 | 0.1295 | 0.6954 |
| ELEKTRIM | 2 | 2 | 0.9087 | 0.1004 | 0.2507 | 0.8609 |
| ELEKTRIM | 2 | 5 | 0.2698 | 0.7476 | 0.1463 | 0.6918 |
| ELEKTRIM | 2 | 6 | 0.1001 | 0.9089 | 0.1235 | 0.3901 |
| IRENA | 2 | 2 | 0.7867 | 0.2290 | 0.2588 | 0.8701 |
| KABLE | 2 | 4 | 0.7757 | 0.2403 | 0.2026 | 0.8035 |
| KROSNO | 2 | 3 | 0.7832 | 0.2324 | 0.2409 | 0.3815 |
| MOSTALEXP | 2 | 2 | 0.8258 | 0.1883 | 0.1189 | 0.8585 |
| MOSTALEXP | 2 | 3 | 0.3340 | 0.6852 | 0.2767 | 0.7522 |
| MOSTALEXP | 2 | 4 | 0.3438 | 0.6757 | 0.7701 | 0.9583 |
| MOSTALEXP | 2 | 5 | 0.3943 | 0.6262 | 0.5050 | 0.8084 |
| MOSTALEXP | 3 | 5 | 0.7314 | 0.2865 | 0.1441 | 0.9184 |
| MOSTALWAR | 2 | 2 | 0.3761 | 0.6455 | 0.4303 | 0.9999 |
| MOSTALWAR | 2 | 3 | 0.3026 | 0.7172 | 0.3173 | 0.6967 |
| OKOCIM | 2 | 3 | 0.8618 | 0.1502 | 0.2574 | 1.0000 |
| OKOCIM | 2 | 4 | 0.4789 | 0.5422 | 0.1003 | 0.9459 |
| POLIFARBC | 2 | 3 | 0.8498 | 0.1637 | 0.7479 | 0.9148 |
| POLIFARBC | 3 | 6 | 0.6422 | 0.3789 | 0.2609 | 0.8462 |
| RAFAKO | 2 | 3 | 0.7622 | 0.2566 | 0.3900 | 0.5852 |
| RAFAKO | 2 | 5 | 0.4763 | 0.5473 | 0.1295 | 0.9203 |
| RAFAKO | 2 | 6 | 0.2471 | 0.7712 | 0.2199 | 0.6669 |
| SOKOLOW | 2 | 3 | 0.7150 | 0.3046 | 0.1162 | 0.3578 |
| SWARZEDZ | 2 | 4 | 0.7763 | 0.2397 | 0.5692 | 0.7880 |
| SWARZEDZ | 2 | 5 | 0.2821 | 0.7353 | 0.2250 | 0.2652 |
| VISTULA | 2 | 2 | 0.7450 | 0.2737 | 0.1613 | 0.7325 |
| WBK | 2 | 4 | 0.4775 | 0.5449 | 0.8423 | 0.7819 |
| WEDEL | 3 | 5 | 0.1288 | 0.8827 | 0.2117 | 0.7716 |