A note on modelling asset returns on the Stock Exchange in Warsaw using hidden Markov and Markov-switching models

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Abstract

Hidden Markov models and Markov-switching models are estimated for 49 long-listed stocks quoted on the Stock Exchange in Warsaw. A test is performed for each model to compare empirical autocorrelation functions of absolute and squared returns and those of the model. On the basis of test results one can state that the two kinds of models fail to explain the behaviour of asset returns on the Warsaw Stock Exchange.

1 Introduction

In 1995 Granger and Ding [6] [7] formulated some stylized facts for daily return series (X_t). One of their temporal properties was as follows [9]:

> TP2: The autocorrelation functions of $|X_t|$ and X_t^2 decay slowly starting from the first autocorrelation, and $\text{corr}(|X_t|, |X_{t-k}|) > \text{corr}(X_t^2, X_{t-k}^2)$. The decay is much slower than the exponential rate of a stationary AR(1) or ARMA$(1, q)$ model. The autocorrelations remain positive for very long lags.

In 1998 Rydén, Teräsvirta and Åsbrink [9] applied the hidden Markov model to subsets of the daily S&P 500 series and concluded that the only stylized fact that cannot be reproduced by this model is TP2. In 2006 Bulla and Bulla [3] introduced hidden semi-Markov models and compared them to the hidden Markov models using 18 series of daily European sector indices. They stated that hidden semi-Markov models reproduce the shape of empirical autocorrelation function of X_t^2 much better than hidden Markov models. In 2004 Bialkowski [2] used the hidden Markov models with two and three states to study monthly returns of six European indices, including the Warsaw Stock Exchange WIG, but did not investigate TP2.

It is a purpose of this paper to study if the property TP2 could be reproduced by hidden Markov models or Markov-switching models in case of asset returns quoted on the Stock Exchange in Warsaw.

2 Methods

It is assumed that a hidden Markov model is a discrete process

$$(C_t, X_t)_{t=1}^{\infty}, \tag{1}$$

where (C_t) is an m-state homogeneous Markov chain with initial state distribution δ and transition matrix Γ and (X_t) is a sequence of independent random variables such that the conditional distribution of X_t depends only on C_t [4, p. 1]. If, given $C_t = i$, X_t is distributed $\mathrm{N}(\mu_i, \sigma_i^2)$, the hidden Markov model is called a normal hidden Markov model [4, p. 13] and may be written as

$$X_t = \mu_{C_t} + \sigma_{C_t}\varepsilon_t, \tag{2}$$

where (ε_t) is a sequence of independent standard normally distributed variables.

If the conditional distribution of X_t depends not only on C_t but also on $X_{t-1}, \ldots, X_{t-k}$, the process (1) is called a Markov-switching model [4, p. 5]. If, given $C_t = i, X_{t-1}, \ldots, X_{t-k}$, X_t is distributed $\mathrm{N}(0, \sigma_i^2)$, the Markov-switching model may be written as

$$X_t = \alpha_{C_t,1}X_{t-1} + \ldots + \alpha_{C_t,k}X_{t-k} + \sigma_{C_t}\varepsilon_t. \tag{3}$$

Having an observation sequence $x_1, \ldots, x_T$ from the model (2), the likelihood of the sample is [11, p. 37]

$$L(\delta, \Gamma, \theta_1, \ldots, \theta_m) = \delta P_1 \Gamma P_2 \Gamma P_3 \ldots \Gamma P_T \mathbf{1}^T,$$

where $\theta_i = (\mu_i, \sigma_i)$, P_t is a diagonal matrix with the i-th diagonal element

$$p_{it} = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left\{-\frac{(x_t - \mu_i)^2}{2\sigma_i^2}\right\}$$

and $\mathbf{1}$ is a row vector of ones. The likelihood can be maximized by the Baum-Welch algorithm [1].

For the model (3), the likelihood function for a sample $x_{1-k}, \ldots, x_T$ is [10, p. 322]

$$L(\delta, \Gamma, \alpha_1, \ldots, \alpha_m, \sigma_1, \ldots, \sigma_m) = \delta\Gamma P_1 \Gamma P_2 \ldots \Gamma P_T \mathbf{1}^T,$$

where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ik})$ and P_t is a diagonal matrix with the i-th diagonal element

$$p_{it} = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_i^2}\left(x_t - \alpha_{i1}x_{t-1} - \ldots - \alpha_{ik}x_{t-k}\right)^2\right\}.$$

The likelihood was maximized by the BOBYQUA algorithm of M. J. D. Powell [8].[1]

The data set consists of 49 longest-listed stocks quoted on the Stock Exchange in Warsaw between April 1991 and June 2010. Depending on the series, from 3062 to 4334 observations were used for estimation. The process of preparing data is described in more detail in [5].

3 Results

For each series of returns, six models have been estimated: three models (2) for $m = 2, 3, 4$ and three models (3) for $m = 2, 3, 4$ and $k = 1$. All respective Markov chains proved to be irreducible, aperiodic and possessing unique stationary distributions. From now on, estimated parameters with δ replaced by the stationary distributions are treated as true parameters.

Figures 1 and 2 picture autocorrelation functions of absolute and squared returns of empirical data and models (2) and (3) with $m = 4$ for the first three stocks. For models (2) autocorrelations can be found analytically [11, p. 34]. For models (3) the Monte Carlo method has been used. The decay of empirical autocorrelation functions is distinctly extremely slow as opposed to autocorrelation functions of the models.

Let $(\rho_i)_{i=1}^n$ and $(r_i)_{i=1}^n$ denote autocorrelation function of absolute or squared returns of an investigated model and that of a series generated by the model, respectively. The distance between (r_i) and (ρ_i) can be measured by

$$d_p = \left(\sum_{i=1}^{n} |r_i - \rho_i|^p \right)^{1/p}, \quad p = 1, 2 \quad \text{and} \quad d_\infty = \max_{i=1,\ldots,n} |r_i - \rho_i|. \tag{4}$$

Let (r_i^0) be the empirical autocorrelation function of absolute or squared returns and let d_p^0 be the distance between (r_i^0) and (ρ_i). A Monte Carlo approximation to the probability $P(d_p \geq d_p^0)$ can be used to evaluate adequacy of the model. Each Monte Carlo experiment was based on 10000 generated series.

Tables 1 and 2 present all these approximations to $P(d_p \geq d_p^0)$, $p = 1, 2, \infty$, which simultaneously proved to be not less than 0,01. As can be seen in table 1, VISTULA is the only stock whose autocorrelation function of absolute returns can be well approximated by a hidden Markov model but this model fails to approximate autocorrelation function of squared returns. The same may be noticed on Markov-switching models.

[1] A C++ translation of Steven G. Johnson from *The NLopt nonlinear-optimization package*, http://ab-initio.mit.edu/nlopt, was used.

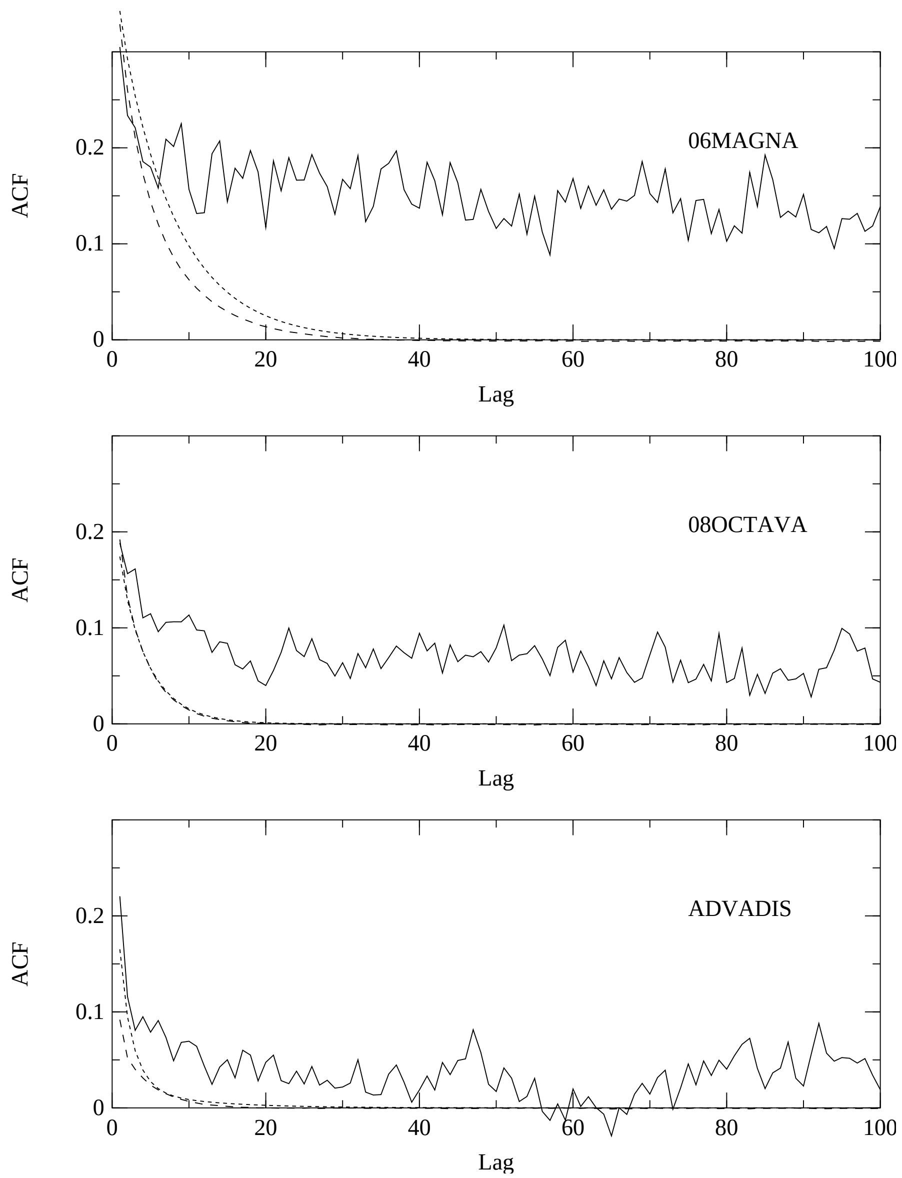

Figure 1: Autocorrelation functions of absolute returns for the first three stocks: empirical (solid line), estimated normal hidden Markov model (2) with four states (dashed line) and estimated Markov-switching model (3) with four states and one lagged return (dotted line).

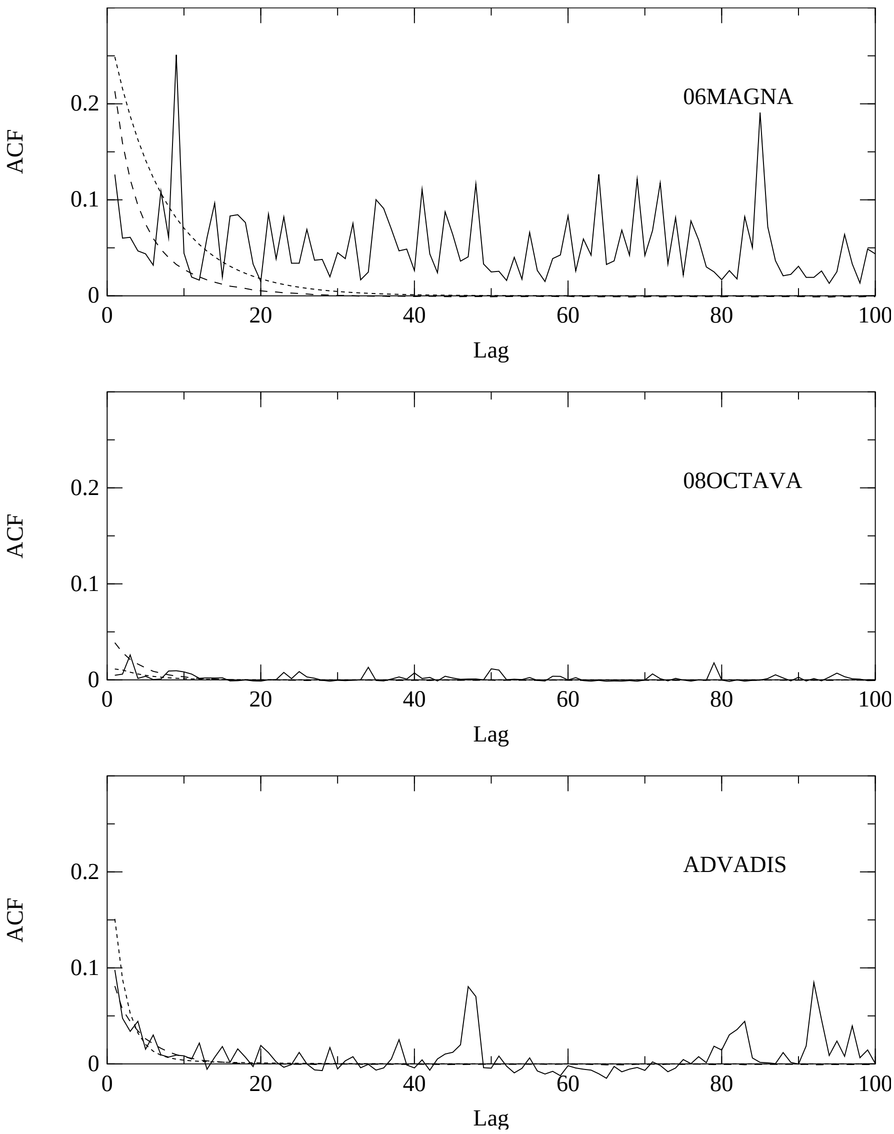

Figure 2: Autocorrelation functions of squared returns for the first three stocks: empirical (solid line), estimated normal hidden Markov model (2) with four states (dashed line) and estimated Markov-switching model (3) with four states and one lagged return (dotted line).

Table 1: Monte Carlo approximations to $P(d_p \geq d_p^0)$ not less than 0,01 for normal hidden Markov models.

Stock	$P(d_1 \geq d_1^0)$	$P(d_2 \geq d_2^0)$	$P(d_\infty \geq d_\infty^0)$
	$m = 3$, absolute returns		
VISTULA	0,0188	0,0177	0,0357
	$m = 2$, squared returns		
ADVADIS	0,9774	0,1774	0,0288
BORYSZEW	1,0000	1,0000	0,4125
NFIEMF	1,0000	1,0000	0,5151
WILBO	1,0000	0,9786	0,0432
	$m = 3$, squared returns		
ADVADIS	0,9783	0,2200	0,0559
BANKBPH	0,8938	0,8384	0,7032
BBICAPNFI	0,7514	0,8060	0,9191
BBIZENNFI	0,9985	0,2466	0,0308
BORYSZEW	0,7926	0,7991	0,8271
IGROUP	0,0828	0,0195	0,0128
MIDAS	0,4056	0,0798	0,1482
NFIEMF	1,0000	1,0000	0,6972
WILBO	0,9981	0,7563	0,2869
	$m = 4$, squared returns		
08OCTAVA	0,9234	0,8438	0,8574
ADVADIS	0,8541	0,3023	0,3103
ALMA	0,0174	0,0224	0,0787
BANKBPH	0,9080	0,8649	0,7127
BBICAPNFI	0,8450	0,8530	0,9344
BBIDEVNFI	0,0196	0,0299	0,0197
BBIZENNFI	0,9965	0,3492	0,0604
BORYSZEW	0,7752	0,8179	0,7830
ECHO	0,4814	0,3878	0,3346
JUPITER	0,1505	0,0220	0,0295
KRUSZWICA	0,0195	0,0183	0,0646
MIDAS	0,3719	0,1453	0,2321
MOSTALEXP	0,0697	0,0714	0,1435
NFIEMF	0,9806	0,9732	0,9090
RELPOL	0,2986	0,3776	0,5315
RUBICON	0,6749	0,4614	0,4118
SWIECIE	0,3548	0,0775	0,0252
WILBO	0,9740	0,8839	0,4426

Table 2: Monte Carlo approximations to $P(d_p \geq d_p^0)$ not less than 0,01 for Markov-switching models.

Stock	$P(d_1 \geq d_1^0)$	$P(d_2 \geq d_2^0)$	$P(d_\infty \geq d_\infty^0)$
	$m = 3$, absolute returns		
VISTULA	0,0884	0,0801	0,0799
	$m = 2$, squared returns		
ADVADIS	0,9770	0,1783	0,0316
BORYSZEW	1,0000	0,9999	0,4035
NFIEMF	1,0000	1,0000	0,5355
WILBO	1,0000	0,9892	0,0392
	$m = 3$, squared returns		
BORYSZEW	0,8254	0,7873	0,7715
IGROUP	0,0808	0,0258	0,0200
KREZUS	0,0103	0,0207	0,2432
MIDAS	0,3836	0,0937	0,1851
NFIEMF	1,0000	1,0000	0,6993
WILBO	0,9990	0,7239	0,2339
	$m = 4$, squared returns		
08OCTAVA	0,8662	0,8033	0,6370
APATOR	0,1185	0,0548	0,0563
BANKBPH	0,9362	0,8248	0,6465
BBICAPNFI	0,9004	0,2651	0,0613
BBIDEVNFI	0,0237	0,0402	0,0331
BORYSZEW	0,6995	0,7344	0,8634
ECHO	0,5722	0,4021	0,3423
IGROUP	0,0303	0,0241	0,0435
KREZUS	0,0163	0,0324	0,2875
MIDAS	0,3366	0,0937	0,1899
NFIEMF	1,0000	1,0000	0,9082
RAFAKO	0,6197	0,1278	0,0296
RELPOL	0,3448	0,3607	0,4558
RUBICON	0,3334	0,1160	0,0550
SWIECIE	0,9904	0,8234	0,1743
WILBO	0,7675	0,7831	0,4832

4 Conclusion

As it was pointed out in [3, sec. 3], the limitation of models based on Markov chains is geometric distribution of sojourn time in a single state. Namely, if $C_t = i$, the probability that (C_t) remains in the state i until the moment $t + k$ is $\gamma_{ii}^k(1 - \gamma_{ii})$.

From the results of tests it would seem that both hidden Markov models and Markov-switching models in the forms we considered have failed to explain the behaviour of asset returns on the Warsaw Stock Exchange.

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